COHERENT STATES FOR THE REGULARIZED HYDROGEN ATOM

MACIEJ HOROWSKI
Institute of Physics, Warsaw University Division,
15-424 Białystok, Lipowa 41, Poland

Odzijewicz[1] has shown that any mechanical system can be described by a triple $(M, \mathcal{M}, \mathcal{K}\colon M \to \mathbf{CP}(\mathcal{M}))$, where M is a differentiable manifold, $\mathcal{M}$ is a complex Hilbert space and $\mathcal{K}$ is a symplectic embedding (in the sense that the pull-back $\mathcal{K}^*\omega_{FS}$ of the Fubini-Study form ω_{FS} on $\mathbf{CP}(\mathcal{M})$ is nondegenerate). According to Odzijewicz,[1] the symplectic manifold $(M, \mathcal{K}^*\omega_{FS})$ is interpreted as the classical phase space of the system. The symplectic manifold $(\mathbf{CP}(\mathcal{M}), \omega_{FS})$ is the space of pure quantum states of the system and the states in $\mathcal{K}(M)$ are interpreted as coherent states. Assuming that the system is in an equilibrium state we can express the action functional, the transition amplitude between coherent states, the Schrödinger equation propagator and other characteristics of the system in terms of the mapping $\mathcal{K}$. [1,2]

This paper presents the results of applying this formalism to the case of the regularized Kepler problem (regularized hydrogen atom). Details and proofs are contained in another paper.[3]

As a phase space we choose, following Rawnsley,[4] the complex manifold

$$\mathbf{C}_3 = \left\{ z \in \mathbf{C}^4 : z \cdot z = z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0,\ z \neq 0 \right\}.$$

Since $\mathbf{C}_3$ is the zero level set of a nondegenerate quadratic form in $\mathbf{C}^4 \setminus \{0\}$, it follows[5] that its image under the natural projection onto $\mathbf{CP}(3)$ is isomorphic to $\mathbf{CP}(1) \times \mathbf{CP}(1)$. Thus $\mathbf{C}_3$ is a $\mathbf{C}^*$ bundle over $\mathbf{CP}(1) \times \mathbf{CP}(1)$. Let $\mathbf{F}^{k,l} \stackrel{\text{def}}{=} pr_1^*(\overset{k}{\bigotimes} \mathbf{E}) \otimes pr_2^*(\overset{l}{\bigotimes} \mathbf{E})$, where pr_i denotes the projection of $\mathbf{CP}(1) \times \mathbf{CP}(1)$ onto the ith factor, $\mathbf{E} \to \mathbf{CP}(1)$ is the universal bundle and $k, l \in \mathbf{Z}$. It is easy to see that

$$\mathbf{C}_3 \cong \mathbf{F}^{1,1} \setminus \{\text{zero section}\}.$$

The Hilbert space $\mathcal{M}$ will be constructed as the space of square integrable (with respect to the Liouville measure) holomorphic sections of some line bundle over $\mathbf{C}_3$. To do this let us consider a family of line bundles $\mathbf{E}^{k,l} \stackrel{\text{def}}{=} \pi^*(\mathbf{F}^{k,l})$, where $\pi\colon \mathbf{C}_3 \to \mathbf{CP}(1) \times \mathbf{CP}(1)$.

The natural action of the group $SU(2)$ on the universal bundle $\mathbf{E} \to \mathbf{CP}(1)$ induces an action of $SU(2) \times SU(2)$ on the bundles $\mathbf{F}^{k,l}$ and $\mathbf{E}^{k,l}$; in particular we obtain an action of this group on $\mathbf{C}_3$.