ON THE DECOMPOSITION OF THE OSCILLATOR REPRESENTATION

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Following Howe [7] we shall call the oscillator representation in $\mathbf{R}^d$ the representation of th Lie algebra $\int\updownarrow(\in, \mathbf{R})$ on the Schwartz space $\mathcal{S}(\mathbf{R}^d)$ or the corresponding (i.e. exponentiated) unita representation on $L^2(\mathbf{R}^d)$ of the double cover $\widetilde{SL}(2,\mathbf{R})$ of $SL(2,\mathbf{R})$ obtained in the following wa Let

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e^+ = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be a standard basis of the Lie algebra $sl(2,\mathbf{R})$ satisfying the commutation relations

$$[h, e^{\pm}] = \pm 2e^{\pm}, \qquad [e^+, e^-] = h.$$

We let $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ stand for the Laplace operator and we shall use the familiar notation $r^2 = r^2(x) = x \cdot x$ for the function on $\mathbf{R}^d$ equal to the square of the radius (for the standard euclidea metric). If D is a differential operator (e.g. the Laplacian) and f a smooth function, we shall wri $D \circ f$ for the composition of D with the operator of multiplication by f. The oscillator representatio ω^d of the Lie algebra $sl(2,\mathbf{R})$ is defined by setting

$$\begin{aligned} \omega^d(h) &= \sum_{j=1}^{d} x_j \frac{\partial}{\partial x_j} + \frac{d}{2}, \\ \omega^d(e^+) &= \frac{i}{2}\sum_{j=1}^{d} x_j^2 = \frac{i}{2} r^2, \\ \omega^d(e^-) &= \frac{i}{2}\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} = \frac{i}{2}\Delta \end{aligned} \qquad (1$$

and extending by linearity to the whole of $sl(2,\mathbf{R})$. The fact that this representation of the L algebra $sl(2,\mathbf{R})$ is a derived representation of a unitary representation of the double cover $\widetilde{SL}(2,\mathbf{R}$ of $SL(2,\mathbf{R})$ is a part of a theorem of Shale and A. Weil asserting the existence of the metaplect representation, cf. e.g. [6] or [3].

Many properties of the oscillator representation are easilly accessed through the use another basis statisfying the standard commutation relations, namely the one given by the matrice

$$k = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad n^+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad n^- = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

which satisfy

$$[k, n^{\pm}] = \pm 2n^{\pm}, \qquad [n^+, n^-] = k.$$