ON THE SPECTRUM OF THE GEODESIC FLOW ON SPHERES

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Abstract

We propose a uniform method for derivation of the energy spectrum of the geodesic flow of the sphere S^n (and hence of the Kepler problem) for all dimensions $n \geq 1$. The idea is to use Marsden-Weinstein reduction in the context of equivariant cohomology. The one-dimensional case is thus covered by the general geometric quantization scheme.

In the present note we propose a general procedure which produces the spectrum (with the multiplicities) of the geodesic flow on the n-dimensional sphere. In view of previous work of many authors,[1-6] the point is to include in the "geometric" quantization scheme[7] the case $n = 1$. We recall that this problem is equivalent with the problem of quantization of the n-dimensional hydrogen atom. Because of limited space we do not reproduce here all proofs and computations, which shall be given elsewhere. The authors are convinced, that the trick introduced (using equivariant instead of ordinary cohomology) should work in several other important cases, and see the treatment of the geodesic flow of S^n bellow as a useful example.

The geodesic flow on S^n is the Hamiltonian system (P, σ, F) where,

$$\begin{aligned} &P = T^*S^n = \{(\xi, \eta) \in R^{n+1} \times R^{n+1};\ |\xi| = 1,\ \langle \xi, \eta \rangle = 0\} \\ &\sigma = d\eta \wedge d\xi, \quad \Phi = |\xi|^2|\eta|^2/2, \qquad n = 1, 2, 3, \ldots \end{aligned} \tag{1}$$

The orbits of these Hamiltonian systems are the great circles on the respective spheres. The energy hypersurfaces $\Phi = \epsilon$ (fixed velocities) can be easily identified with the Stiefel manifolds of oriented orthonormal two-frames in R^{n+1},

$$V(2, n+1) = SO(n+1)/SO(n-1),$$

thus

$$P_{1/2} = \Phi^{-1}(1/2) = V(2, n+1)$$

and

$$P_\epsilon \cong P_{1/2} \qquad (\epsilon \neq 0)$$

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